

INTEGRAL FORMULAS FOR SUBMANIFOLDS AND THEIR APPLICATIONS

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Introduction

Integral formulas of Minkowski type have been studied and applied in characterizing umbilical submanifolds by Chen [3], Katsurada [5], [6], [7], Kôjyô [6], Nagai [7], Okumara [10], Tani [11] and Yano [3], [8], [9], [10], [11]. These authors assumed that the normal vector field e with respect to which the integral formulas were obtained was parallel in the normal bundle¹. The purpose of this paper is to extend the study of the above authors. We obtain the most general integral formulas for a submanifold of a Riemannian space of constant sectional curvature without putting any restriction on the unit normal vector field e , and under conditions which are weaker than the condition that e be parallel in the normal bundle we obtain integral formulas of Minkowski type and apply them to the study of umbilical submanifolds. We give concrete illustrations to substantiate our generalisations.

1. Preliminaries

Let M be an orientable differentiable manifold of dimension n imbedded in an orientable m -dimensional Riemannian manifold N of constant sectional curvature. Let $u^a = u^a(x^h)$ denote the local expression of the submanifold M in N . Here and in the sequel a, b, c, \dots run over the range $1, 2, \dots, m$, and h, i, j, \dots over the range $1, 2, \dots, n$ unless otherwise specified. We shall identify vector fields of M with their images under the differential mapping. Thus if X is a vector field of M and has local expression $X = X^h \partial_h$, then it has local expression $X = X^h B_h^a \partial_a$ in N where $\partial_h = \partial/\partial x^h$, $\partial_a = \partial/\partial u^a$, $B_h^a = \partial u^a / \partial x^h$, and Einstein's summation convention is followed for repeated

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¹ For a generalization of the results of these authors see C. C. Hsiung, J. D. Liu and S. S. Mitra, *Integral formulas for closed submanifolds of a Riemannian manifold*, J. Differential Geometry **12** (1977) 133-151, which was published after the present paper had been written.

indices. If G denotes the Riemannian metric of N and G_{ab} its components, the components g_{ij} of the induced Riemannian metric g of M are given by $g_{ij} = G_{ab}B_i^aB_j^b$. Let $\bar{\nabla}$ and ∇ denote the Riemannian connections of M and N respectively; they are related by the Gauss formula:

$$(1.1) \quad \bar{\nabla}_X Y = \nabla_X Y + H(X, Y),$$

where X, Y are vector fields of M , and H is the second fundamental form of M .

Let e be the unit normal vector field on M . The Weingarten formula for M is given by

$$(1.2) \quad \bar{\nabla}_X e = -A_e(X) + \nabla_X^\perp e,$$

where the Weingarten map A_e is related to the second fundamental form H by

$$(1.3) \quad g(A_e(X), Y) = G(H(X, Y), e)$$

for all vector fields X, Y of M . Let e_1, \dots, e_{m-n} form an orthonormal basis in the normal bundle of M , and h^x be the second fundamental form corresponding to e_x so that

$$(1.4) \quad H(X, Y) = h^x(X, Y)e_x,$$

where and in the sequel x, y run over the range $1, 2, \dots, m-n$. The local expression for the equation of Codazzi is

$$(1.5) \quad \nabla_k h_{ji}^x - \nabla_j h_{ki}^x = h_{ji}^y l_{ky}^x - h_{ki}^y l_{jy}^x,$$

where

$$(1.6) \quad \nabla_j^\perp e_y = l_{jy}^x e_x \quad \text{with } l_{jy}^x = -l_{jx}^y.$$

Let e be a unit normal vector field on M . We set $e = e_1$ and choose the other normals e_2, \dots, e_{m-n} in such a way that

$$\det(\partial_1, \dots, \partial_n, e_1, \dots, e_{m-n}) = 1.$$

Set $h^1 = h$, $A_{e_1} = A$ and $l_{iy}^1 = l_{iy}$ for convenience. Denote the principal curvatures of M with respect to e by k_1, \dots, k_n . Define s_0, s_1, \dots, s_n ; p_0, p_1, \dots, p_n by

$$(1.7) \quad s_0 = 1, \quad s_1 = \sum_{i_1 < \dots < i_1} k_{i_1} \dots k_{i_1},$$

$$(1.8) \quad p_0 = 1, \quad p_l = \sum_i (k_i)^l,$$

and the l th mean curvature M_l by

$$(1.9) \quad M_0 = 1, \quad \binom{n}{l} M_l = s_l,$$

where $\binom{n}{l}$ are binomial coefficients and $l = 1, 2, \dots, n$. It is easy to see that

$$(1.10) \quad p_l = h_{i_2}^{i_1} h_{i_3}^{i_2} \dots h_{i_l}^{i_{l-1}}.$$

The p_i 's and the s_i 's are related by Newton's formulas:

$$(1.11) \quad p_l - s_1 p_{l-1} + \dots + (-1)^{l-1} s_{l-1} p_1 + (-1)^l l s_l = 0,$$

where $l = 1, 2, \dots, n$. s_l can be solved in terms of p_1, \dots, p_n , and we have

$$(1.12) \quad s_l = \sum_{\substack{t_1+2t_2+\dots+t_l=l \\ 0 \leq t_i}} \frac{(-1)^{t_1+t_2+\dots+t_l}}{(t_1)! \dots (t_n)! 2^{t_2} \dots l^{t_l}} p_1^{t_1} \dots p_l^{t_l}.$$

2. Integral formulas

Let \bar{Y} be a vector field of N defined along M . We may write $\bar{Y} = Z + p^y e_y$, where Z is a vector field tangential to M and $p^y = G(\bar{Y}, e_y)$. We call p^y the support function with respect to e_y . We have

$$(2.1) \quad \bar{\nabla}_i \bar{Y} = (\nabla_i Z^j - p^y h_{iy}^j) \partial_j + (Z^j h_{ij}^y + p^x l_{ix}^y + \nabla_i p^y) e_y.$$

We denote the tangential component of $\bar{\nabla}_i \bar{Y}$ by $\tan \bar{\nabla}_i \bar{Y}$ so that

$$(2.2) \quad \nabla_i Z^j = p^y h_{iy}^j + g(\tan \bar{\nabla}_i \bar{Y}, \partial_k) g^{kj}.$$

Consequently

$$(2.3) \quad \nabla_i Z^i = p p_1 + p^z h_{iz}^i + g(\tan \bar{\nabla}_i \bar{Y}, \partial_k) g^{ki},$$

where we have set $p^1 = p$ and allowed z to run over the range $2, \dots, m - n$. Also the normal component of $\bar{\nabla}_i \bar{Y}$ is given by

$$(2.4) \quad G(\bar{\nabla}_i \bar{Y}, e_x) = Z^j h_{ij}^x + \nabla_i p^x + p^y l_{iy}^x.$$

Define $h_{(l)i}^j, Z_{(l)}^j$ for $l = 0, 1, \dots, n$ by

$$(2.5) \quad \begin{aligned} h_{(0)i}^j &= \delta_i^j, & h_{(l)i}^j &= h_{i_1}^j h_{i_2}^{i_1} \dots h_{i_l}^{i_{l-1}}, \\ Z_{(0)}^j &= Z^j, & Z_{(l)}^j &= h_{(l)i}^j Z^i. \end{aligned}$$

Using (1.5), (1.10) and (2.5) we obtain

$$(2.6) \quad \begin{aligned} \nabla_i Z_{(l)}^i &= (\nabla_i p_1) Z_{(l-1)}^i + \frac{1}{2} (\nabla_i p_2) Z_{(l-2)}^i + \dots + \frac{1}{l} (\nabla_i p_l) Z^i \\ &+ p p_{l+1} + h_{(l)}^{ij} \left\{ g(\tan \bar{\nabla}_j \bar{Y}, \partial_i) + p^z h_{jiz} \right\} + Z_{(l-1)k}^i D_{ij}^k, \end{aligned}$$

where we have set

$$(2.7) \quad Z_{(l-1)k}{}^{ij} = \delta_k^i Z_{(l-1)}^j + h_k^i Z_{(l-2)}^j + \cdots + h_{(l-1)k}{}^i Z^j,$$

$$(2.8) \quad D_{ij}{}^k = l_{ij} h_j^{ky} - l_{jy} h_i^{ky}.$$

Let t be a real number. To obtain the main integral formula we compute the following, using (1.12) and (2.6):

$$(2.9.1) \quad \begin{aligned} \nabla_i(p^t Z_{(l)}^i) &= t p^{t-1} (\nabla_i p) Z_{(l)}^i + p^t [(\nabla_i p_1) Z_{(l-1)}^i \\ &+ \frac{1}{2} (\nabla_i p_2) Z_{(l-2)}^i + \cdots + \frac{1}{l} (\nabla_i p_l) Z^i + p p_{l-1} \\ &+ h_{(l)}{}^{ji} \{ g(\tan \bar{\nabla}_j \bar{Y}, \partial_i) + p^z h_{jiz} \} + Z_{(l-1)k}{}^{ij} D_{ij}{}^k], \end{aligned}$$

$$(2.9.2) \quad \begin{aligned} \nabla_i(p^t s_1 Z_{(l-1)}^i) &= p_1 t p^{t-1} (\nabla_i p) Z_{(l-1)}^i + p^t [(\nabla_i p_1) Z_{(l-1)}^i + p_1 \{ (\nabla_i p_1) Z_{(l-2)}^i \\ &+ \frac{1}{2} (\nabla_i p_2) Z_{(l-3)}^i + \cdots + \frac{1}{(l-1)} (\nabla_i p_{l-1}) Z^i + p p_l \\ &+ h_{(l-1)}{}^{ji} \{ g(\tan \bar{\nabla}_j \bar{Y}, \partial_i) + p^z h_{jiz} \} + Z_{(l-2)k}{}^{ij} D_{ij}{}^k], \end{aligned}$$

$$(2.9.3) \quad \begin{aligned} \nabla_i(p^t s_2 Z_{(l-2)}^i) &= t p^{t-1} (\nabla_i p) \frac{1}{2} (p_1^2 - p_2) Z_{(l-2)}^i + p^t \left[\left\{ p_1 (\nabla_i p_1) - \frac{1}{2} \nabla_i p_2 \right\} Z_{(l-2)}^i \right. \\ &+ \frac{1}{2} (p_1^2 - p_2) \left\{ (\nabla_i p_1) Z_{(l-3)}^i + \frac{1}{2} (\nabla_i p_2) Z_{(l-4)}^i + \cdots \right. \\ &+ \left. \frac{1}{l-2} (\nabla_i p_{l-2}) Z^i + p p_{l-1} + h_{(l-2)}{}^{ji} \{ g(\tan \bar{\nabla}_j \bar{Y}, \partial_i) + p^z h_{jiz} \} \right. \\ &\left. \left. + Z_{(l-3)k}{}^{ij} D_{ij}{}^k \right\} \right], \end{aligned}$$

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$$(2.9.l+1) \quad \begin{aligned} \nabla_i(p^t s_l Z^i) &= \sum_{\substack{t_1+2t_2+\dots \\ +l_t=l}} \frac{(-1)^{t_1+\dots+t_{l+1}}}{(t_1!) \cdots (t_l!) 2^{t_2} \cdots l^{t_l}} \left[t p^{t-1} (\nabla_i p) p_1^{t_1} \cdots p_l^{t_l} Z^i \right. \\ &+ p^t \{ \nabla_i (p_1^{t_1} \cdots p_l^{t_l}) Z^i \\ &\left. + p_1^{t_1} \cdots p_l^{t_l} (p p_1 + g(\tan \bar{\nabla}_i \bar{Y}, \partial_k) g^{ki} + p^z h_{iz}{}^i) \} \right]. \end{aligned}$$

Suppose the submanifold M of N has closed regular boundary B_{n-1} . Integrating (2.9.1)–(2.9.2) + \cdots + $(-1)^l (2.9.l+1)$ over M and making use

of (1.9), (1.11) and Stoke's theorem we get

$$\begin{aligned}
 & \int_{B_{n-1}} p^t \sum_i (-1)^{i-1} |g|^{\frac{1}{2}} \{ Z_{(l)}^i - s_1 Z_{(l-1)}^i + \dots + (-1)^l s_l Z^i \} \\
 & \quad \cdot dx^1 \wedge \dots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \dots \wedge dx^n \\
 (2.10) \quad & = \int_M (-1)^l (l+1) \binom{n}{l+1} p^{t+1} M_{l+1} dV + \int_M \{ H_l(e) + C_l(e) \} dV \\
 & \quad + \int_M p^t \{ h_{(l)}^{ji} - s_1 h_{(l-1)}^{ji} + \dots + (-1)^l s_l h^{ji} \} g(\tan \bar{\nabla}_j \bar{Y}, \partial_i) dV \\
 & \quad + \int_M p^{t-1} \{ Z_{(l)}^i - s_1 Z_{(l-1)}^i + \dots + (-1)^l s_l Z^i \} (\nabla_i p) dV,
 \end{aligned}$$

where $|g|$ denotes the determinant of the matrix $((g_{ij}))$, dV denotes the volume element of M , and

$$(2.11) \quad H_l(e) = p^2 h^j{}_z \{ h_{(l)ji} - s_1 h_{(l-1)ji} + \dots + (-1)^l s_l g_{ij} \},$$

$$(2.12) \quad C_l(e) = \{ Z_{(l-1)k}^{ij} - s_1 Z_{(l-2)k}^{ij} + \dots + (-1)^{l-1} s_{l-1} Z^j \delta_k^i \} D_{ij}^k,$$

$l = 1, 2, \dots, n - 1$, and for convenience we define $C_0(e) = 0$.

Lemma 2.1. *The invariant $C_l(e)$ defined by (2.12) is zero for all $l = 1, 2, \dots, n - 1$, if*

$$(2.13) \quad G(\bar{\nabla}_X Y, \nabla_Z^\perp e) = G(\bar{\nabla}_Z Y, \nabla_X^\perp e)$$

for all vector fields X, Y, Z of M . In particular, $C_l(e) = 0$ if e is parallel in the normal bundle.

Proof. Suppose (2.13) holds. Setting $X = \partial_j, Y = \partial_k, Z = \partial_i$ and using (1.1) we have

$$G(\bar{\nabla}_X Y, \nabla_Z^\perp e) = G\left(h_{jk}^x e_x - \sum_{y=1}^{m-n} l_{iy} e_y\right) = -h_{jk}^y l_{iy}.$$

Hence (2.13) implies that $D_{ij}^k = l_{iy} h_j^{ky} - l_{iy} h_i^{ky} = 0$, which in view of (2.12) implies that $C_l(e) = 0$. q.e.d.

Thus the condition that $C_l(e) = 0$ for some $l, 1 \leq l \leq n - 1$, is weaker than the condition that e is parallel in the normal bundle. However, when $m - n = 2$ the condition (2.13) is equivalent to the condition that e is parallel in the normal bundle provided at least two principal curvatures with respect to e_2 never vanish on M .

Lemma 2.2. *The invariant $H_l(e)$ defined by (2.11) can be expressed in the form*

$$(2.14) \quad H_l(e) = \frac{(-1)^l}{l!} \sum p^2 h_{jz} k_i k_{i_2} \dots k_{i_l},$$

where the summation is taken over all the distinct indices j, i_1, \dots, i_l .

Proof. Since $H_l(e)$ is an invariant we can use the frame of principal vectors v_1, v_2, \dots, v_n of M with respect to e to evaluate it. Thus since $h_{(l)}^{ij} = (k_j)^l \delta^{ij}$ (no summation with respect to j), from (2.11) we have

$$(2.15) \quad H_l(e) = \sum_{j=1}^n p^z h_{jjz} \{ (k_j)^l - s_1(k_j)^{l-1} + \dots + (-1)^l s_l \}.$$

But (see [1, Lemma 1.1])

$$(k_j)^l - s_1(k_j)^{l-1} + \dots + (-1)^l s_l = (-1)^l \sum_{\substack{i_1 < \dots < i_l \\ i_1, \dots, i_l \neq j}} k_{i_1} \dots k_{i_l}.$$

So substituting in (2.15) we obtain (2.14).

Remark 2.1. (2.10) is the most general integral formula for a submanifold M of a Riemannian manifold N of constant sectional curvature. If M is a hypersurface of N , then $p^z = 0$, $z = 2, 3, \dots, m - n$, $l_y = 0$, and (2.10) together with the formula (2.4) for $\nabla_i p$ reduces to the integral formula obtained earlier by Amur and Hegde [2].

We shall discuss other special cases in §3.

3. Applications of the integral formulas: characterizations of umbilical submanifolds

We consider applications of the integral formulas (2.10) in obtaining various characterizations of umbilical submanifolds under the hypothesis that \bar{Y} is some special vector field such as a concurrent vector field, a conformal Killing vector field etc., and that there is a unit normal vector field e on M satisfying the conditions $H_l(e) = C_l(e) = 0$ for some l , $0 \leq l \leq n - 1$, where $H_l(e)$ and $C_l(e)$ are invariants given by (2.11) and (2.12) respectively.

Throughout the following discussion we shall assume that M is a closed submanifold of a Riemannian manifold N of constant sectional curvature and that the real number $t = 0$.

3.1. The case where \bar{Y} is a concurrent vector field.

In the first instance we obtain integral formulas of Minkowski type from (2.10), and then use them to characterize umbilical submanifolds.

Since we have assumed that the vector field \bar{Y} of N defined along M is a concurrent vector field, we have

$$(3.1) \quad \bar{\nabla}_j \bar{Y} + \partial_j = 0.$$

Hence

$$\begin{aligned}
 (3.2) \quad & \{h_{(l)}^{ij} - s_1 h_{(l-1)}^{ij} + \dots + (-1)^l s_l g^{ij}\} g(\tan \bar{\nabla}_j \bar{Y}, \partial_i) \\
 & = -\{p_l - s_1 p_{l-1} + \dots + (-1)^l n s_l\} \\
 & = (-1)^{l-1} (l+1) \binom{n}{l+1} M_l,
 \end{aligned}$$

by virtue of (1.9) and (1.11). Observing that M is closed and $t = 0$ and substituting (3.2) in (2.10) we have

Proposition 3.1. *Let N admit a concurrent vector field \bar{Y} along M , and let e be a unit normal vector field on M . Then*

$$(3.3) \quad (-1)^l (l+1) \binom{n}{l+1} \int_M (M_l - p M_{l+1}) dV = \int_M \{H_l(e) + C_l(e)\} dV,$$

$l = 0, 1, \dots, n-1,$

where M_l is the l th mean curvature with respect to e , $H_l(e)$ and $C_l(e)$ are invariants given by (2.11) and (2.12) respectively, and p is the support function with respect to e .

As immediate consequences of the above proposition we have the following theorems.

Theorem 3.2. *Let N admit a concurrent vector field \bar{Y} along M . If e is a unit normal vector field on M such that $H_l(e) = C_l(e) = 0$ for some $l, 0 \leq l \leq n-1$, then*

$$(3.4) \quad \int_M (M_l - p M_{l+1}) dV = 0.$$

Theorem 3.3. *Let N admit a concurrent vector field \bar{Y} along M . If $p^z A_{e_z} = 0$, and e is unit normal field on M such that $C_l(e) = 0$ for some $l, 0 \leq l < n-1$, then*

$$(3.5) \quad \int_M (M_l - p M_{l+1}) dV = 0.$$

Proof. The condition $p^z A_{e_z} = 0$ implies that $p^z h_{ijz} = 0$ for all i, j , so that from (2.11) we have $H_l(e) = 0$ for all l . The result then follows from Theorem 3.2.

Theorem 3.4. *Let N admit a concurrent vector field \bar{Y} along M . For the unit normal field e on M if $H_l(e) = 0, \nabla_Z^\perp e = 0$, where Z is the component of \bar{Y} tangential to M , and if M is umbilical with respect to each of the normals e_2, \dots, e_{m-n} , then $C_l(e) = 0$ and*

$$(3.6) \quad \int_M (M_1 - p M_2) dV = 0.$$

Proof. Since M is umbilical with respect to each of the normals e_2, \dots, e_{m-n} , we have

$$(3.7) \quad h_{ji}^z = k^z g_{ji}, \quad z = 2, \dots, m - n,$$

where k^z are real valued functions on M . Now using (3.7) in (2.12) we get

$$\begin{aligned} C_l(e) &= Z^j \delta_k^i k^z (l_{iz} \delta_j^k - l_{jz} \delta_i^k) \\ &= (1 - n) k^z Z^j l_{jz} \\ &= (n - 1) k^z G(\nabla_Z^\perp e, e_2). \end{aligned}$$

Thus if $\nabla_Z^\perp e = 0$, then $C_l(e) = 0$, and from Theorem 3.1 we get (3.6). *q.e.d.*

In the above theorem, if we replace the condition $\nabla_Z^\perp e = 0$ by

$$\nabla_Z^\perp e = \nabla_{Z_{(1)}}^\perp e = \nabla_{Z_{(2)}}^\perp e = \dots = \nabla_{Z_{(l-1)}}^\perp e = 0,$$

where

$$Z_{(1)} = A_e(Z), \dots, Z_{(l-1)} = A_e(Z_{(l-2)}),$$

then with computations analogous to those in Theorem 3.4 it can be shown that $C_j(e) = 0, j = 1, 2, \dots, l$. Thus we have

Theorem 3.5. *Let N admit a concurrent vector field \bar{Y} along M , and let e be a unit normal vector field on M . If there is an integer $l, 0 < l < n$, such that*

- (i) $H_l(e) = 0$,
 - (ii) $\nabla_Z^\perp e = 0, \nabla_{Z_{(i)}}^\perp e = 0, \quad i = 1, 2, \dots, l - 1$,
- where Z is the component of \bar{Y} tangential to M , and

(iii) M is umbilical with respect to each of the normal e_2, \dots, e_{m-n} , then $C_k(e) = 0, k = 1, 2, \dots, l$, and

$$\int_M (M_l - p M_{l+1}) dV = 0.$$

Remarks 3.1. (a) If M is a hypersurface of N , then clearly $H_l(e) = C_l(e) = 0$ for all l , and (3.3) yields Minkowski-Hsiung formulas for M , [4].

(b) Suppose $N = E^m$, and X is the position vector field of M in E^m with respect to the origin of E^m . Since we can identify $\bar{\nabla}_i X$ with ∂_i , it is clear that we can set $\bar{Y} = -X$. Let X_n be the normal part of X . Set $e = e_1$, and choose the other normals e_2, \dots, e_{m-n} in such a way that e_2 is in the direction of $X_n - (X_n \cdot e_1)e_1$. Then clearly the support functions p^3, \dots, p^{m-n} are all zero and $p^1 = -(X \cdot e_1), p^2 = -(X \cdot e_2)$. Thus from (2.14) we have

$$H_l(e) = (-1)^{l+1} \sum (X \cdot e_2) h_{ij}^2 k_i \dots k_j,$$

where the summation is taken over all distinct indices $j, i_1 \cdots i_l = 1, 2, \dots, n$. Set

$$(3.8) \quad F_{l+1}(e) = \frac{(-1)^{l+1}}{l+1} \binom{n}{l+1}^{-1} H_l(e).$$

Further let e be parallel in the normal bundle. By Lemma 2.1 it follows that $C_l(e) = 0$ for all l . The formula (3.3) then becomes

$$(3.9) \quad \int_M \{M_l + (X \cdot e)M_{l+1} + F_{l+1}(e)\} dV = 0, \quad l = 0, 1, \dots, n-1.$$

This formula was obtained by Chen and Yano [3] by a different procedure which involves the use of vector forms. We have not only generalized the above equations suitably, but also explicitly shown how the results of Chen and Yano are related to ours.

(c) With the assumptions as in (b), Theorems 3.2 and 3.3 reduce to those obtained by Chen and Yano [3].

(d) In Theorems 3.4 and 3.5 we find concrete illustrations of the fact that a condition weaker than the condition that e be parallel in the normal bundle can be used to make $C_l(e) = 0$ for some $l, 0 < l \leq n-1$. For, the condition $\nabla_{\frac{1}{Z}} e = 0$ in Theorem 3.4 or the condition $\nabla_{\frac{1}{Z}} e = \nabla_{Z(0)}^{\perp} e = \cdots = \nabla_{Z^{(l-1)}}^{\perp} e = 0$ in Theorem 3.5 are clearly weaker than the condition that e be parallel in the normal bundle.

We need the following well-known lemmas for proving results on umbilicity of M .

Lemma 3.6. *Let $M_l, l = 0, 1, \dots, n$, be as in (1.9). Then*

$$M_l^2 - M_{l-1}M_{l+1} \geq 0,$$

and further equality in (3.7) implies that M is umbilical with respect to the unit normal vector field e .

Lemma 3.7. *For integers l, s such that $0 \leq l < s \leq n$, if M_l, M_{l+1}, \dots, M_s are positive, then*

$$\frac{M_l}{M_{l+1}} \leq \frac{M_{l+1}}{M_{l+2}} \leq \dots \leq \frac{M_{s-1}}{M_s},$$

and equality at any stage implies that M is umbilical with respect to e .

Lemma 3.8 (Chen and Yano [3]). *For integers l, s such that $1 \leq l < s \leq n$, if M_1, \dots, M_s are positive and there are constants $c_j (l \leq j \leq s-1)$ such that $M_s = \sum_{j=l}^{s-1} c_j M_j$, then*

$$M_{s-1} - \sum_{j=l}^{s-1} c_j M_{j-1} \geq 0,$$

where the equality holds only if M is umbilical with respect to e .

The following theorems which give characterization of the umbilicity of M are extension of those proved by Chen and Yano [3]. We sketch the proofs briefly and for details we refer to [3].

Theorem 3.9. *If there are a unit normal vector field e on M and an integer l , $1 \leq l < n$, such that*

- (i) $M_{l+1} > 0$,
- (ii) $p \geq M_l/M_{l+1}$ (or $p \leq M_l/M_{l+1}$),
- (iii) $H_j(e) = 0$, $C_j(e) = 0$, $j = l - 1, l$,

then M is umbilical with respect to e .

Proof. By (ii) and Theorem 3.2 we have

$$p = \frac{M_l}{M_{l+1}} \text{ and } \int_M (M_{l-1} - pM_l)dV = 0,$$

which together with (i) imply $M_{l+1}M_{l-1} - M_l^2 = 0$. Thus by Lemma 3.6 M is umbilical with respect to e .

Theorem 3.10. *If there are a unit normal vector field e on M and an integer, l , $1 < l < n$, such that*

- (i) $M_{l-1}, M_l, M_{l+1} > 0$,
- (ii) $p \leq M_{l-1}/M_l$,
- (iii) $H_l(e) = 0$, $C_l(e) = 0$,

then M is umbilical with respect to e .

Proof. By Theorem 3.2 and (iii) we have $\int_M (M_l - pM_{l+1})dV = 0$ and by (ii) and Lemma 3.7 it follows that $p \leq M_{l-1}/M_l \leq M_l/M_{l+1}$. These results together imply $p \leq M_{l-1}/M_l \leq M_l/M_{l+1} = p$. So by Lemma 3.6, M is umbilical with respect to e .

Theorem 3.11. *If there are a unit normal vector field e on M and integers l , s , $1 \leq l < s \leq n$, such that*

- (i) M_l, M_{l+1}, \dots, M_s are positive,
- (ii) $M_s = \sum_{j=l}^{s-1} c_j M_j$ for some constants $c_j \geq 0$, $l \leq j \leq s$,
- (iii) $H_j(e) = 0$, $C_j(e) = 0$, $j = l - 1, \dots, s - 2$,

then M is umbilical with respect to e .

Proof. Proof follows from Theorem 3.2 and Lemma 3.8.

Theorem 3.12. *If there are a unit normal vector field e on M such that*

- (i) $M_n, M_{n-1} > 0$,
- (ii) the sum $\sum_{i=1}^n 1/k_i$ of principal radii of curvatures of M with respect to e is constant,

then M is umbilical with respect to e .

Proof. Follows from Theorem 3.11 and the fact that $\sum_{i=1}^n 1/k_i = n M_{n-1}/M_n = \text{constant}$.

Theorem 3.13. *If there are a unit normal vector field e on M and an integer l , $1 \leq l < n$, such that*

- (i) M_l, M_{l+1} are constants,
- (ii) $H_j(e) = 0, C_j(e) = 0, j = l - 1, l$,

then M is umbilical with respect to e .

Proof. By (i) and Theorem 3.2 we have

$$\int_M p \, dV = \frac{M_l}{M_{l+1}} \int_M dV = \frac{1}{M_l} \int_M M_{l-1} \, dV,$$

which implies $\int_M (M_{l+1}M_{l-1} - M_l^2) dV = 0$, and hence from Lemma 3.6 it follows that M is umbilical with respect to e .

Theorem 3.14. *If there is a unit normal vector field e on M such that*

- (i) $M_1 = \text{constant}$,
- (ii) $H_0(e) = p^z h_{jz}^j = 0, H_1(e) = 0, C_1(e) = 0$,
- (iii) p keeps the same sign on M ,

then M is umbilical with respect to e .

Proof. By (i) and Theorem 3.2 we have

$$\int_M dV = M_1 \int_M p \, dV = \frac{1}{M_1} \int_M p \, M_2 \, dV,$$

which implies $\int_M p(M_1^2 - M_2) \, dV = 0$. If p keeps the same sign on M , then $M_1^2 - M_2 = 0$ which by Lemma 3.6 implies that M is umbilical with respect to e .

Theorem 3.15. *If there is a unit normal vector field e on M such that*

- (i) $H_0(e) = 0, H_1(e) = 0$,
- (ii) $\nabla_{\frac{1}{Z}} e = 0$ where Z is the tangential part of \bar{Y} ,
- (iii) M is umbilical with respect to orthonormal vector fields e_2, \dots, e_{m-n} where e, e_2, \dots, e_{m-n} form an orthonormal basis of the normal bundle,
- (iv) $M_1 = \text{constant}$,
- (v) p keeps the same sign on M ,

then M is a totally umbilical submanifold of N .

Proof. The result follows from Theorems 3.4 and 3.14.

Remark 3.2. If the mean curvature vector of the submanifold M of a Euclidean space E^m (resp. a sphere S^m in E^{m+1}) is assumed to be parallel in the normal bundle of M in $E^m(S^m)$, it can be shown that M_1 is constant [9]. Further if the mean curvature vector is assumed to be in the direction of the first normal e and $p^z h_{jz} = 0$ for all i, j , then $H_0(e) = 0, H_1(e) = 0$ and $C_1(e) = 0$. Hence Theorem 3.14 generalizes the following theorems due to Yano [9].

Theorem A. *Suppose that the mean curvature vector of a compact orientable submanifold M of a Euclidean space E^m does not vanish, and we take the first*

unit normal e_1 to M in the direction of the mean curvature vector. If the mean curvature vector is parallel with respect to the connection induced in the normal bundle of M in E^m , $p^2 h_{jiz} = 0$ and p has a fixed sign, then the submanifold lies on a sphere S^{m-1} .

Theorem B. Suppose that the mean curvature vector of a compact orientable submanifold M of a sphere S^{m-1} does not vanish, and we take the first unit normal e_1 to M in the direction of the mean curvature vector. If the mean curvature vector is parallel with respect to the connection induced in the normal bundle of M in S^{m-1} , $p^2 h_{jiz} = 0$ and p has a fixed sign, then the submanifold lies on a sphere S^{m-2} .

3.2. The case where \bar{Y} is a conformal Killing vector field.

Since

$$g(\tan \bar{\nabla}_j \bar{Y}, \partial_k) = B_j^b B_k^a \bar{\nabla}_b \bar{Y}_a,$$

where $\bar{Y}_a = G_{ab} \bar{Y}^b$ we have,
(3.10)

$$\begin{aligned} & \{h_{(l)}^{kj} - s_1 h_{(l-1)}^{kj} + \dots + (-1)^l s_l g^{kj}\} g(\tan \bar{\nabla}_j \bar{Y}, \partial_k) \\ &= \frac{1}{2} \{h_{(l)}^{kj} - s_1 h_{(l-1)}^{kj} + \dots + (-1)^l s_l g^{kj}\} B_k^b B_j^a L_{\bar{Y}} G_{ba}, \end{aligned}$$

where $L_{\bar{Y}} G_{ba}$ is the Lie derivative of the metric tensor G_{ba} with respect to \bar{Y} . Throughout this part we assume that \bar{Y} is a conformal Killing vector field so that

$$(3.11) \quad L_{\bar{Y}} G_{ab} = 2\rho G_{ab},$$

where ρ is a function. Substituting from (3.10) and (3.11) in (2.10) and observing that M is closed and compact and that $t = 0$ we get

$$(3.12) \quad (-1)^l (l+1) \binom{n}{l+1} \int_M (\rho M_l + p M_{l+1}) dV = \int_M \{H_l(e) + C_l(e)\} dV, \\ l = 0, 1, \dots, n-1.$$

As immediate consequences of this integral formula we have

Theorem 3.16. Let N admit a conformal Killing vector field \bar{Y} along M satisfying (3.11). If there are a unit normal vector field e on M and an integer l , $0 \leq l < n$, such that $H_l(e) = 0$, $C_l(e) = 0$, then

$$(3.13) \quad \int_M (\rho M_l + p M_{l+1}) dV = 0.$$

Theorem 3.17. Let N admit a conformal Killing vector field \bar{Y} along M satisfying (3.11). If there is a unit normal vector field e on M such that the

normal component of \bar{Y} is parallel to e and $C_l(e) = 0$ for some $l, 0 \leq l < n$, then $H_l(e) = 0$ and

$$(3.14) \quad \int_M (\rho M_l + p M_{l+1}) dV = 0.$$

Theorem 3.18. *Let N admit a conformal Killing vector field \bar{Y} along M satisfying (3.11). If there is a unit normal field e such that $H_l(e) = 0$ and $\nabla_Z^\perp e = 0$, where Z is the component of \bar{Y} tangential to M , and if M is umbilical with respect to each of normal fields, e_2, \dots, e_{m-n} , where e, e_2, \dots, e_{m-n} is an orthonormal frame in the normal bundle, then $C_l(e) = 0$ and*

$$\int_M (\rho M_1 + p M_2) dV = 0.$$

From Theorems 3.16, 3.17 and 3.18 we have the following results on the umbilicity of M .

Theorem 3.19. *If e is a unit normal vector field on M such that*

- (i) $M_1 = \text{constant}$,
 - (ii) $C_1(e) = 0, H_0(e) = H_1(e) = 0$,
 - (iii) p keeps the same sign on M ,
- then M is umbilical with respect to e .

Proof. By Theorem 3.16 and (ii) we have

$$\int_M (\rho + p M_1) dV = 0, \quad \int_M (\rho M_1 + p M_2) dV = 0,$$

which together with (i) yield

$$\int_M p(M_1^2 - M_2) dV = 0.$$

From (iii) and Lemma 3.7 it follows that $M_1^2 - M_2 = 0$, and hence M is umbilical with respect to e .

Theorem 3.20. *If e is a unit normal vector field on M , and there is an integer $l, 0 < l < n$, such that*

- (i) $M_l = \text{constant}$,
 - (ii) M_1, \dots, M_{l+1} are positive,
 - (iii) $H_l(e) = 0, C_l(e) = 0, H_0(e) = 0$,
 - (iv) p keeps the same sign on M ,
- then M is umbilical with respect to e .

Proof. By (i) and Theorem 3.16 we have

$$\int_M (\rho M_l + p M_1 M_l) dV = 0, \quad \int_M (\rho M_l + p M_{l+1}) dV = 0,$$

which yield

$$\int_M p(M_1 M_l - M_{l+1}) dV = 0.$$

From (iv) and Lemma 3.7 it follows that $M_1 M_l - M_{l+1} = 0$, and hence M is umbilical with respect to e .

Theorem 3.21. *If e is a unit normal vector field on M such that*

- (i) $\rho + pM_1 \geq 0$ (or $\rho + pM_1 \leq 0$),
- (ii) $C_1(e) = 0, H_0(e) = 0, H_1(e) = 0$,
- (iii) p keeps the same sign on M ,

then M is umbilical with respect to e .

Proof. By Theorem 3.16, (i) and (ii) we have $\rho + pM_1 = 0$ and $\int_M (\rho M_1 + pM_2) dV = 0$. Hence

$$\int_M p(M_1^2 - M_2) dV = 0.$$

Consequently by (iii) and Lemma 3.6 we get the desired result.

Theorem 3.22. *If e is a unit normal vector field on M , and there is an integer $l, 0 < l < n$, such that*

- (i) $M_l > 0$,
- (ii) $p > -M_{l-1}/M_l$ (or $p \leq -M_{l-1}/M_l$),
- (iii) $H_j(e) = 0, C_j(e) = 0, j = l-1, l$,
- (iv) ρ keeps the same sign on M ,

then M is umbilical with respect to e .

Proof. By (ii), (iii) and Theorem 3.16 we get

$$\int \frac{\rho}{M_l} (M_l^2 - M_{l-1} M_{l+1}) dV = 0.$$

Hence from Lemma 3.6 and (iv) it follows that M is umbilical with respect to e .

Theorem 3.23. *If e is a unit vector field on M , and there is an integer $l, 0 < l < n$, such that*

- (i) $M_{l-1}, M_l, M_{l+1} > 0$,
- (ii) $p \geq -\rho M_{l-1}/M_l$,
- (iii) $H_l(e) = 0, C_l(e) = 0$,
- (iv) ρ is positive on M ,

then M is umbilical with respect to e .

Proof. The result follows from Theorem 3.16 and Lemma 3.7.

Theorem 3.24. *If e is a unit normal field on M such that*

- (i) M_1 is constant,
- (ii) $H_0(e) = 0, H_1(e) = 0$,
- (iii) $\nabla_{\frac{1}{Z}} e = 0$, where Z is the tangential part of \bar{Y} ,
- (iv) p keeps the same sign on M ,

(v) M is umbilical with respect to each of normal fields e_2, \dots, e_{m-n} , where e, e_2, \dots, e_{m-n} form an orthonormal frame in the normal bundle, then M is totally umbilical submanifold of N .

Proof. It is a consequence of Theorems 3.18 and 3.19.

Theorem 3.25. *If e is a unit normal vector field on M such that*

- (i) $\rho + pM \geq 0$,
 - (ii) $H_0(e) = 0, H_1(e) = 0$.
 - (iii) $\nabla_Z^{\perp} e = 0$, where Z is the tangential part of \bar{Y} ,
 - (iv) p keeps the same sign on M ,
 - (v) M is umbilical with respect to each of normal fields e_2, \dots, e_{m-n} , where e, e_2, \dots, e_{m-n} form an orthonormal basis in the normal bundle,
- then M is totally umbilical submanifold of N .

Proof. The result follows from Theorems 3.16 and 3.21.

Remark 3.3. Suppose e is a unit vector field in the direction of mean curvature vector of M , and the conformal Killing vector field \bar{Y} is such that its normal part is parallel to e . Further suppose that e is parallel in the normal bundle. Then $C_l(e) = 0, H_l(e) = 0$ for all l . With this setup theorems analogous to Theorems 3.19 to 3.21 were obtained by Katsurada and Kojyo [6], Katsurada and Nagai [7].

3.3. The case where \bar{Y} is a concircular vector field.

Throughout this part \bar{Y} is assumed to be a concircular vector field, that is, \bar{Y} satisfies

$$(3.15) \quad \bar{\nabla}_b \bar{Y}_a = \rho G_{ba} + \bar{X}_b \bar{Y}_a,$$

where ρ is a function, and \bar{X}_a are the components of a 1-form associated with a gradient vector field \bar{X} of N defined along M . We have

$$(3.16) \quad L_{\bar{Y}} G_{ba} = 2\rho G_{ba} + \bar{X}_b \bar{Y}_a + \bar{X}_a \bar{Y}_b.$$

Setting $B_j^b \bar{X}_b = X_j$ and substituting (3.16) and (3.10) and using (1.10), (1.11) and (1.9) we get

$$(3.17) \quad \{h_{(l)}^{kj} - s_1 h_{(l-1)}^{kj} + \dots + (-1)^l s_l g^{ji}\} g(\tan \bar{\nabla}_j \bar{Y}, \partial_k) \\ = (-1)^{l+1} (l+1) \binom{n}{l+1} \rho M_l + \{Z_{(l)}^i - s_1 Z_{(l-1)}^i + \dots + (-1)^l s_l Z^i\} X_i.$$

Substituting from (3.17) in (2.10) and observing that M is closed and compact and that $t = 0$ we get

$$(3.18) \quad (-1)^{l+1} (l+1) \binom{n}{l+1} \int_M (\rho M_l + p M_{l+1}) dV \\ = \int_M \{H_l(e) + C_l(e) + K_l(e)\} dV,$$

where

$$(3.19) \quad K_l(e) = \{Z_{(l)}^i - s_1 Z_{(l-1)}^i + \cdots + (-1)^l s_l Z^i\} X_i.$$

In view of (3.18) we have

Theorem 3.26. *Let N admit a concircular vector field \bar{Y} along M satisfying (3.15). If there are a unit normal vector field e on M and an integer l , $0 \leq l < n$, such that $H_l(e) = 0$, $C_l(e) = 0$ and $K_l(e) = 0$, then*

$$(3.20) \quad \int_M (\rho M_l + p M_{l+1}) dV = 0.$$

The proofs of the following theorems each of which gives a characterization of an umbilical submanifold are similar to those of §3.2. Hence we simply state the theorems.

Theorem 3.27. *If e is a unit normal vector field on M such that*

- (i) $M_1 = \text{constant}$,
- (ii) $H_j(e) = C_j(e) = K_j(e) = 0$, $j = 0, 1$,
- (iii) p keeps the same sign on M ,

then M is umbilical with respect to e .

Theorem 3.28. *If e is a unit normal vector field on M , and there is an integer l , $0 < l < n$, such that*

- (i) M_l is constant,
- (ii) M_1, \dots, M_{l+1} are positive,
- (iii) $H_l(e) = K_l(e) = C_l(e) = 0$, $H_0(e) = 0$,
- (iv) p keeps the same sign on M ,

then M is umbilical with respect to e .

Theorem 3.29. *If e is a unit normal vector field on M such that*

- (i) $\rho + p M_1 \geq 0$ (or $\rho + p M_1 \leq 0$),
- (ii) $C_l(e) = H_l(e) = K_l(e) = 0$, $l = 0, 1$,
- (iii) p keeps the same sign on M ,

then M is umbilical with respect to e .

Theorem 3.30. *If e is a unit normal vector field on M , and there is an integer l , $0 < l < n$, such that*

- (i) $M_{l-1}, M_l, M_{l+1} > 0$,
- (ii) $p \geq -\rho M_{l-1} / M_l$,
- (iii) $H_l(e) = 0$, $C_l(e) = 0$, $K_l(e) = 0$,
- (iv) ρ is positive on M ,

then M is umbilical with respect to e .

Remark 3.4. If the ambient space is Euclidean m -space E^m , and M is imbedded into a hypersphere of E^m centered at C , then M is said to be a spherical submanifold or simply spherical, $X - C$ is called the radius vector

field where X is the position vector field of M in E^m with respect to the origin of E^m . Chen and Yano [3] proved the following theorem.

Theorem C. *If M is imbedded in E^m , then there exists a normal vector field $e \neq 0$ over M such that*

- (1) *e is parallel in the normal bundle, and*
- (2) *M is umbilical with respect to e ,*

when and only when M is spherical, and e is parallel to the radius vector field.

By taking $N = E^m$ and assuming that the unit normal vector field e is parallel in the normal bundle, the conclusion in each of the theorems, in §3, on umbilicity of M with respect to e can be replaced by “ M is spherical, and e is parallel to the radius vector field”.

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